

PERFECT PARTITION OF SOME REGULAR BIPARTITE GRAPHS

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ABSTRACT. A graph has a perfect partition if all its perfect matchings can be partitioned so that each part is a 1-factorization of the graph. Let $L_{rm,r} = K_{rm,rm} - mK_{r,r}$. We first give a formula to count the number of perfect matchings of $L_{rm,r}$, then show that $L_{6,1}$ and $L_{8,2}$ have perfect partitions.

1. INTRODUCTION

It is well-known that every regular bipartite graph has a perfect matching, and furthermore, every perfect matching of a regular bipartite graph is in a 1-factorization, that is, a collection of pairwise perfect matchings whose union is the original graph. Here we study the inverse problem: given the perfect matchings of a graph, can one partition them into 1-factorizations?

This perfect partition problem was introduced in [1], in the language of matrices. Let S_n be the set of $n \times n$ permutation matrices, i.e., $(0, 1)$ -matrix each of whose row and column contains exactly one 1. Let A be an $n \times n$ $(0, 1)$ -matrix, a permutation matrix P is contained in A , denoted by $P < A$, if $A - P$ has nonnegative entries, and we let

$$\mathcal{S}(A) = \{P \in S_n : P < A\}.$$

We say that a $(0, 1)$ -matrix has a perfect partition if $\mathcal{S}(A)$ can be partitioned into subsets so that the sum of permutation matrices in each subset is A .

One can see that the two definitions in the preceding paragraphs are equivalent. For a regular bipartite graph G , let the *associated matrix* $A(G)$ of G be the adjacency matrix so that the rows and columns are the two parts of G . Thus a perfect matching in G is a permutation matrix in the associated matrix $A(G)$, and a 1-factorization of G is a set of permutation matrices whose sum is $A(G)$.

It is not hard to construct bipartite graphs or square matrices which have no perfect partition.

Example 1.1. Let P be the permutation matrix corresponding to permutation (12345) in the cycle representation, and let $A = I_5 + P + P^2$. Then every 1-factorization of the graph $G(A)$ associated with A should contain exactly three perfect matchings, but $G(A)$ contains 13 perfect matchings.

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The perfect partition problem is interesting for graphs in which every perfect matching is in a 1-factorization. Because of this, we consider a special kind of regular bipartite graphs which contain many perfect matchings.

For $r, m \in \mathbb{N}$, let $L_{rm,r} = K_{rm,rm} - mK_{r,r}$. Denote by J_n the $n \times n$ matrices with all entries equal to 1. In terms of matrices, that is the matrix obtained from $J_{rm \times rm}$ by replacing the ones on the $r \times r$ disjoint submatrices on the main diagonal by zeros. Evidently, a permutation matrix P satisfies $P < A(L_{n,1})$ if and only if it is a derangement, i.e., a permutation matrix with zero diagonal entries.

In [1], the authors showed that the number of perfect matchings in G equals to the permanent of the associated matrix $A(G)$, and that the permanent of $A(L_{rm,r})$ is a multiple of $r(m-1)$ using the Laplace expansion formula for permanent, see [[2], p199]. Thus $L_{rm,r}$ satisfies the easy necessary condition to have a perfect partition. Here we give a formula to calculate the number of perfect matchings in $L_{rm,r}$, which generalizes the formula for the number of derangements [[2], p202]: $D_n = n! \sum_{k=0}^n (-1)^k \frac{1}{k!}$.

Theorem 1.2. *Let $n = rm$. Then the number of perfect matchings in $L_{n,r}$ is*

$$|M(L_{n,r})| = a_0 n! - a_1 (n-1)! + \cdots + (-1)^n a_n 0! = n! \sum_{k=0}^n \frac{a_k}{k!},$$

where

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \left(\sum_{k=0}^r k! \binom{r}{k}^2 x^k \right)^m$$

Proof. View $L_{n,r}$ as a chessboard with forbidden positions at zeroes in the matrix, we can think of a permutation matrix (perfect matching) to be a way to n non-attacking rooks so that none of the rooks are in those forbidden positions. Let a_i be the number of ways to place n non-attacking rooks so that at least m rooks are in the forbidden positions, then by the Principle of Inclusion and Exclusion,

$$|M(L_{n,r})| = \sum_{i=0}^n (-1)^i a_i (n-i)!.$$

Let $A(x) = \sum_i a_i x^i$ be the rook polynomial. Then $A(x) = B(x)^m$, where $B(x) = \sum_{k=0}^r b_k x^k$ is the rook polynomial to place non-attacking rooks in the $r \times r$ matrix, since the ways to place rooks in the r square matrices do not interfere with each other. Thus we have the theorem. \square

For some special m and r , it is not hard to show $L_{rm,r}$ has a perfect partition. For example, let $G = K_{n,n}$ and $A(G)$ be the corresponding matrix of G . Then every permutation matrix of $A(G)$ corresponds to an element in S_n . Let P be a permutation of order n , then the subgroup H generated by P gives a 1-factorization of G . The left cosets of H give a perfect partition of G . So, we have the following, see [1].

Theorem 1.3. *The complete bipartite graphs $K_{n,n}$ (or $L_{n,0}$) has a perfect partition.*

By using the cosets of subgroups, it is also shown in [1] that the set A_n of even permutations has a perfect partition and $L_{2n,n}$ has a perfect partition. However, it becomes hard to solve when the permutations in a given matrix have no group structures. In [1], the authors obtained the following.

Theorem 1.4. *The graphs $L_{4,1}$, $L_{5,1}$ and $L_{6,2}$ have perfect partitions.*

It is challenging to find a perfect partition for $L_{n,1}$ when $n \geq 6$. Note that we need to find a perfect partition of the derangements of n elements. In [1], five different strategies were proposed to show that $L_{6,1}$ has a perfect partition, but none of them led to a solution. In this paper, we use a different strategy to show that $L_{6,1}$ indeed has a perfect partition. It is easy enough to list a perfect partition of $L_{6,1}$ into 53 sets with 5 perfect matchings each. Nevertheless, we will give a theoretical proof in Section 2, and hope that the proof techniques can inspire future study on $L_{n,1}$ for $n \geq 7$. (Note that it is not so easy to list 792 sets with 6 perfect matchings in the case of $L_{7,1}$.) Also, we will show that the graph $L_{8,2}$ has a perfect partition in Section 3. Our construction for $L_{8,2}$ used the perfect partition of $L_{4,1}$ from [1]. Again, we hope that the techniques can inspire future advance of the partition problem.

2. PERFECT PARTITION OF $L_{6,1}$

The purpose of this section is to prove the following.

Theorem 2.1. *The graph $L_{6,1}$ has a perfect partition.*

We divide the proofs of Theorem 2.1 into several lemmas. For notational convenience, we will not distinguish between the graph $L_{rm,r}$ and its adjacency matrix $A(L_{rm,r})$.

By Theorem 1.2, $L_{6,1}$ consists of the 265 derangements in S_6 . According to their cycle structure, we see the following.

Lemma 2.2. *Let T be the derangements with one 6-cycle, C_{33} be the derangements with a product of two 3-cycles, C_{24} be the derangements with a product of a 2-cycle and 4-cycle, and among those, C_{24}^0 be the ones with 1 in the 2-cycle, and let C_{222} be the derangements with a product of three 2-cycles. Then $\mathcal{S}(L_{6,1}) = C_6 \cup C_{24} \cup C_{33} \cup C_{222}$ and furthermore,*

$$|C_6| = 120, |C_{33}| = 40, |C_{24}| = 90, |C_{24}^0| = 30, |C_{222}| = 15.$$

Our proof of Theorem 2.1 will use the following partition strategy:

- T_1 : 30 subsets consisting of four elements from C_6 and one element from C_{24}^0 ;
- T_2 : 16 subsets consisting of three elements from $C_{24} - C_{24}^0$ and two elements from C_{33} ;
- T_3 : 3 subsets consisting of four elements from $C_{24} - C_{24}^0$ and one element from C_{222} ;
- T_4 : 4 subsets consisting of two elements from C_{33} and three elements from C_{222} .

For each $\sigma = (1\ x_2)(x_3\ x_4\ x_5\ x_6) \in C_{24}^0$, let

$$f(\sigma) = \{\sigma, (1\ x_3\ x_2\ x_5\ x_4\ x_6), (1\ x_4\ x_2\ x_6\ x_5\ x_3), (1\ x_5\ x_2\ x_3\ x_6\ x_4), (1\ x_6\ x_2\ x_4\ x_3\ x_5)\}.$$

Lemma 2.3. *The set $T_1 = \{f(\sigma) : \sigma \in C_{24}^0\}$ gives a perfect partition of $C_{24}^0 \cup C_6$.*

Proof. It is easy to see that $f(\sigma)$ is a 1-factorization for any give $\sigma \in C_6$. On the other hand, given $\tau = (1 y_2 y_3 y_4 y_5 y_6) \in C_6$, we find that the only elements in $C_{2,4}^0$ whose images under f contain τ are $(1 y_3)(y_2 y_5 y_4 y_6)$, $(1 y_3)(y_6 y_2 y_5 y_4)$, $(1 y_3)(y_4 y_6 y_2 y_5)$, $(1 y_3)(y_5 y_4 y_6 y_2)$, but these are all the same element. Thus each C_6 element is covered by exactly one $C_{2,4}^0$ element, so that the 30 elements of $C_{2,4}^0$ cover all 120 elements in C_6 . \square

According to above strategy, here is the partition of $C_{24}^0 \cup C_6$.

TABLE 1. 30 sets in T_1 , where each uses 4 from C_6 and 1 from C_{24}^0 .

1	(1 2)(3 4 5 6)	(1 3 2 5 4 6)	(1 4 2 6 5 3)	(1 5 2 3 6 4)	(1 6 2 4 3 5)
2	(1 2)(3 4 6 5)	(1 3 2 6 4 5)	(1 4 2 5 6 3)	(1 6 2 3 5 4)	(1 5 2 4 3 6)
3	(1 2)(3 5 6 4)	(1 3 2 6 5 4)	(1 5 2 4 6 3)	(1 6 2 3 4 5)	(1 4 2 5 3 6)
4	(1 2)(3 5 4 6)	(1 3 2 4 5 6)	(1 5 2 6 4 3)	(1 4 2 3 6 5)	(1 6 2 5 3 4)
5	(1 2)(3 6 5 4)	(1 3 2 5 6 4)	(1 6 2 4 5 3)	(1 5 2 3 4 6)	(1 4 2 6 3 5)
6	(1 2)(3 6 4 5)	(1 3 2 4 6 5)	(1 6 2 5 4 3)	(1 4 2 3 5 6)	(1 5 2 6 3 4)
7	(1 3)(2 4 5 6)	(1 2 3 5 4 6)	(1 4 3 6 5 2)	(1 5 3 2 6 4)	(1 6 3 4 2 5)
8	(1 3)(2 4 6 5)	(1 2 3 6 4 5)	(1 4 3 5 6 2)	(1 6 3 2 5 4)	(1 5 3 4 2 6)
9	(1 3)(2 5 6 4)	(1 2 3 6 5 4)	(1 5 3 4 6 2)	(1 6 3 2 4 5)	(1 4 3 5 2 6)
10	(1 3)(2 5 4 6)	(1 2 3 4 5 6)	(1 5 3 6 4 2)	(1 4 3 2 6 5)	(1 6 3 5 2 4)
11	(1 3)(2 6 5 4)	(1 2 3 5 6 4)	(1 6 3 4 5 2)	(1 5 3 2 4 6)	(1 4 3 6 2 5)
12	(1 3)(2 6 4 5)	(1 2 3 4 6 5)	(1 6 3 5 4 2)	(1 4 3 2 5 6)	(1 5 3 6 2 4)
13	(1 4)(2 3 5 6)	(1 2 4 5 3 6)	(1 3 4 6 5 2)	(1 5 4 2 6 3)	(1 6 4 3 2 5)
14	(1 4)(2 3 6 5)	(1 2 4 6 3 5)	(1 3 4 5 6 2)	(1 6 4 2 5 3)	(1 5 4 3 2 6)
15	(1 4)(2 5 6 3)	(1 2 4 6 5 3)	(1 5 4 3 6 2)	(1 6 4 2 3 5)	(1 3 4 5 2 6)
16	(1 4)(2 5 3 6)	(1 2 4 3 5 6)	(1 5 4 6 3 2)	(1 3 4 2 6 5)	(1 6 4 5 2 3)
17	(1 4)(2 6 5 3)	(1 2 4 5 6 3)	(1 6 4 3 5 2)	(1 5 4 2 3 6)	(1 3 4 6 2 5)
18	(1 4)(2 6 3 5)	(1 2 4 3 6 5)	(1 6 4 5 3 2)	(1 3 4 2 5 6)	(1 5 4 6 2 3)
19	(1 5)(2 3 4 6)	(1 2 5 4 3 6)	(1 3 5 6 4 2)	(1 4 5 2 6 3)	(1 6 5 3 2 4)
20	(1 5)(2 3 6 4)	(1 2 5 6 3 4)	(1 3 5 4 6 2)	(1 6 5 2 4 3)	(1 4 5 3 2 6)
21	(1 5)(2 4 6 3)	(1 2 5 6 4 3)	(1 4 5 3 6 2)	(1 6 5 2 3 4)	(1 3 5 4 2 6)
22	(1 5)(2 4 3 6)	(1 2 5 3 4 6)	(1 4 5 6 3 2)	(1 3 5 2 6 4)	(1 6 5 4 2 3)
23	(1 5)(2 6 4 3)	(1 2 5 4 6 3)	(1 6 5 3 4 2)	(1 4 5 2 3 6)	(1 3 5 6 2 4)
24	(1 5)(2 6 3 4)	(1 2 5 3 6 4)	(1 6 5 4 3 2)	(1 3 5 2 4 6)	(1 4 5 6 2 3)
25	(1 6)(2 3 4 5)	(1 2 6 4 3 5)	(1 3 6 5 4 2)	(1 4 6 2 5 3)	(1 5 6 3 2 4)
26	(1 6)(2 3 5 4)	(1 2 6 5 3 4)	(1 3 6 4 5 2)	(1 5 6 2 4 3)	(1 4 6 3 2 5)
27	(1 6)(2 4 5 3)	(1 2 6 5 4 3)	(1 4 6 3 5 2)	(1 5 6 2 3 4)	(1 3 6 4 2 5)
28	(1 6)(2 4 3 5)	(1 2 6 3 4 5)	(1 4 6 5 3 2)	(1 3 6 2 5 4)	(1 5 6 4 2 3)
29	(1 6)(2 5 4 3)	(1 2 6 4 5 3)	(1 5 6 3 4 2)	(1 4 6 2 3 5)	(1 3 6 5 2 4)
30	(1 6)(2 5 3 4)	(1 2 6 3 5 4)	(1 5 6 4 3 2)	(1 3 6 2 4 5)	(1 4 6 5 2 3)

Definition 2.4. Let $\sigma = (1 x y)(a b c) \in C_{3,3}$ with $a < b, c$. Then the class of σ is y if $b < c$ and x otherwise.

By the definition, a permutation and its inverse have the same class; furthermore, if they are of class y , then one of the them can be written as $(1xy)(abc)$ so that $a < b < c$. In the following of the paper, we will refer σ to be $(1xy)(abc)$ and use σ^{-1} to be the inverse.

Note that if $\sigma = (1xy)(abc)$ has class y , then $\sigma^* = (1yx)(abc)$ and its inverse have class x .

Now suppose $\sigma = (1xy)(abc)$ has class y . We will choose the subset for σ, σ^{-1} so that each $C_{2,4} - C_{2,4}^0$ element contains y in the 2-cycle, then there are two possible ways to finish determining the $C_{2,4} - C_{2,4}^0$ elements:

- (1) $\{(1xy)(abc), (1yx)(acb), (1\bar{a}x\bar{b})(y\bar{c}), (1\bar{c}x\bar{a})(y\bar{b}), (1\bar{b}x\bar{c})(y\bar{a})\}$
- (2) $\{(1xy)(abc), (1yx)(acb), (1\bar{a}x\bar{c})(y\bar{b}), (1\bar{b}x\bar{a})(y\bar{c}), (1\bar{c}x\bar{b})(y\bar{a})\}$

We introduce the concept of a *pattern* to decide the associated subsets with $\sigma \in C_{3,3}$.

Definition 2.5. Let $\sigma = (1xy)(abc)$ be of class y . Then σ and σ^{-1} have pattern $\beta = (1xy)(wvu)$ where $\{w, v, u\} = \{a, b, c\}$ if and only if σ and σ^{-1} are associated with the following three elements in $C_{2,4}$:

$$(1wxv)(yu), (1vxu)(yw), (1uxw)(yv).$$

We will let the pattern for σ^* and $(\sigma^*)^{-1}$ be β^{-1} .

For example, if $\sigma = (1xy)(abc)$ of class y has pattern $(1xy)(abc)$, then σ, σ^{-1} and their associated elements give (1), and $\sigma^*, (\sigma^*)^{-1}$ and their associated elements give (2).

For a given element $\sigma = (1xy)(abc) \in C_{3,3}$ of class y with pattern $\beta = (1xy)(ab'c')$, where $\{b', c'\} = \{b, c\}$, we can define the set $\mathcal{Z}_y^\beta(\sigma)$, the *zone* y which consists of four subsets of five permutations, according to the following rules:

- (i) Determine the three elements in $C_{2,4} - C_{2,4}^0$ associated with σ and σ^{-1} ;
- (ii) Determine the other three pairs of elements in $C_{3,3}$ with class y . By definition, they have the form $\gamma = (1 * y)(***)$ (and γ^{-1}) so that the three elements in the second cycle are in increasing order.
- (iii) Determine the pattern for each γ and γ^{-1} : if γ is associated with $(1uvw)(yk) \in C_{2,4}$ and σ is associated with $(1u'xw')(yk) \in C_{2,4}$, then $(u'xw') = (uvw)$ but $u'xw' \neq uvw$.
- (iv) Write down the elements in $\mathcal{Z}_y^\beta(\sigma)$, which are the four pairs of class y elements together with their associated elements in $C_{2,4} - C_{2,4}^0$.

For example, if we take $\sigma = (123)(465)$ with pattern $(132)(465)$, then we will get zone 2 (note that $(123)(465)$ is of class 2) as follows:

$(1\ 2\ 3)(4\ 6\ 5)$	$(1\ 3\ 2)(4\ 5\ 6)$	$(1\ 6\ 3\ 5)(2\ 4)$	$(1\ 4\ 3\ 6)(2\ 5)$	$(1\ 5\ 3\ 4)(2\ 6)$
$(1\ 2\ 4)(3\ 6\ 5)$	$(1\ 4\ 2)(3\ 5\ 6)$	$(1\ 5\ 4\ 6)(2\ 3)$	$(1\ 6\ 4\ 3)(2\ 5)$	$(1\ 3\ 4\ 5)(2\ 6)$
$(1\ 2\ 5)(3\ 6\ 4)$	$(1\ 5\ 2)(3\ 4\ 6)$	$(1\ 6\ 5\ 4)(2\ 3)$	$(1\ 3\ 5\ 6)(2\ 4)$	$(1\ 4\ 5\ 3)(2\ 6)$
$(1\ 2\ 6)(3\ 5\ 4)$	$(1\ 6\ 2)(3\ 4\ 5)$	$(1\ 4\ 6\ 5)(2\ 3)$	$(1\ 5\ 6\ 3)(2\ 4)$	$(1\ 3\ 6\ 4)(2\ 5)$

Lemma 2.6. For a given $\sigma \in C_{3,3}$ of class $y > 1$ with a given pattern β , \mathcal{Z}_y^β consists of four disjoint subsets of five distinct permutations. Furthermore, if $\sigma, \gamma \in C_{3,3}$ are two elements in the subsets of \mathcal{Z}_y with patterns β_σ and β_γ , respectively, then $\mathcal{Z}_y^{\beta_\sigma}(\sigma) = \mathcal{Z}_y^{\beta_\gamma}(\gamma)$.

Proof. By definition, the three elements in $C_{3,3}$ are determined. Also, once we know the patterns, then the associated elements in $C_{2,4}$ are also determined. So we only need to show that the patterns are well-defined as well. Let $(1uxv)(yk), (1vzk)(yu) \in C_{2,4}$ be associated with σ which are used to determine the pattern of γ . First, $\gamma = (1vy)(***)$. So the corresponding associated elements from $C_{2,4}$ are $(1xvu)(yk)$ and $(1kvx)(yu)$, and the patterns are $(1vy)(xuk)$ and $(1vy)(kxu)$ which are the same.

For the “furthermore” part, we just need to show that σ with pattern β_σ determines γ and its pattern β_γ , then the converse is also true. One can readily verify this statement. \square

Lemma 2.7. For each $z \in \{2, 3, 4, 5, 6\} - \{y\}$, $\mathcal{Z}_y^\beta(\sigma)$ uniquely determines sets \mathcal{Z}_z so that $\mathcal{Z}_z \cap \mathcal{Z}_y^\beta(\sigma) = \emptyset$. Moreover, if $z \neq z'$, then $\mathcal{Z}_z \cap \mathcal{Z}_{z'} = \emptyset$.

Proof. Let $\sigma = (1zy)(abc) \in C_{3,3}$ be in zone y with pattern β , then $\sigma^* = (1zy)(acb)$ of class z is in zone z with pattern β^{-1} , thus $\mathcal{Z}_z = \mathcal{Z}_z^{\beta^{-1}}(\sigma^*)$, by the process described above. By construction and the previous Lemma, the set \mathcal{Z}_z is unique.

Similarly, suppose that $(1z'z)(abc) \in \mathcal{Z}_z$ is of pattern β_1 , then $(1zz')(abc) \in \mathcal{Z}_{z'}$ is of pattern β_1^{-1} , so \mathcal{Z}_z determines $\mathcal{Z}_{z'}$. \square

For example, for the previous σ and the pattern, we could get zone 3 as follows:

(1 2 3)(4 5 6)	(1 3 2)(4 6 5)	(1 5 2 6)(3 4)	(1 6 2 4)(3 5)	(1 4 2 5)(3 6)
(1 3 4)(2 6 5)	(1 4 3)(2 5 6)	(1 6 4 5)(3 2)	(1 2 4 6)(3 5)	(1 5 4 2)(3 6)
(1 3 5)(2 6 4)	(1 5 3)(2 4 6)	(1 4 5 6)(3 2)	(1 6 5 2)(3 4)	(1 2 5 4)(3 6)
(1 3 6)(2 5 4)	(1 6 3)(2 4 5)	(1 5 6 4)(3 2)	(1 2 6 5)(3 4)	(1 4 6 2)(3 5)

The following is the zone 4:

(1 2 4)(3 5 6)	(1 4 2)(3 6 5)	(1 6 2 5)(4 3)	(1 3 2 6)(4 5)	(1 5 2 3)(4 6)
(1 3 4)(2 5 6)	(1 4 3)(2 6 5)	(1 5 3 6)(4 2)	(1 6 3 2)(4 5)	(1 2 3 5)(4 6)
(1 4 5)(2 6 3)	(1 5 4)(2 3 6)	(1 6 5 3)(4 2)	(1 2 5 6)(4 3)	(1 3 5 2)(4 6)
(1 4 6)(2 5 3)	(1 6 4)(2 3 5)	(1 3 6 5)(4 2)	(1 5 6 2)(4 3)	(1 2 6 3)(4 5)

Here is zone 5:

(1 2 5)(3 4 6)	(1 5 2)(3 6 4)	(1 4 2 6)(3 5)	(1 6 2 3)(4 5)	(1 3 2 4)(6 5)
(1 3 5)(2 4 6)	(1 5 3)(2 6 4)	(1 6 3 4)(2 5)	(1 2 3 6)(4 5)	(1 4 3 2)(6 5)
(1 4 5)(2 3 6)	(1 5 4)(2 6 3)	(1 3 4 6)(2 5)	(1 6 4 2)(3 5)	(1 2 4 3)(6 5)
(1 6 5)(2 4 3)	(1 5 6)(2 3 4)	(1 4 6 3)(2 5)	(1 2 6 4)(3 5)	(1 3 6 2)(4 5)

Now zone 6:

(1 2 6)(3 4 5)	(1 6 2)(3 5 4)	(1 5 2 4)(3 6)	(1 3 2 5)(4 6)	(1 4 2 3)(5 6)
(1 3 6)(2 4 5)	(1 6 3)(2 5 4)	(1 4 3 5)(2 6)	(1 5 3 2)(4 6)	(1 2 3 4)(5 6)
(1 4 6)(2 3 5)	(1 6 4)(2 5 3)	(1 5 4 3)(2 6)	(1 2 4 5)(3 6)	(1 3 4 2)(5 6)
(1 5 6)(2 3 4)	(1 6 5)(2 4 3)	(1 3 5 4)(2 6)	(1 4 5 2)(3 6)	(1 2 5 3)(4 6)

For a given $y_0 \in \{2, 3, 4, 5, 6\}$, the elements in $\cup_{y \in [6] - \{1, y_0\}} \mathcal{Z}_y$ form T_2 . We will take the elements in \mathcal{Z}_{y_0} together with elements in $C_{2,2,2}$ to form T_3 and T_4 .

For $\sigma = (1xy_0)(a'b'c') \in C_{3,3}$ of class y_0 with pattern $(1xy_0)(abc)$, we let

$$f(\sigma) = \{\sigma, \sigma^{-1}, (1a)(xb)(y_0c), (1b)(xc)(y_0a), (1c)(xa)(y_0b)\}.$$

Lemma 2.8. *The set $T_4 = \{f(\sigma) : \sigma \in C_{3,3} \text{ and of class } y_0\}$ is a perfect partition of class y_0 elements in $C_{3,3}$ and elements in $C_{2,2,2}$ with no 2-cycle $(1y_0)$.*

Proof. We just need to show that $f(\sigma) \cap f(\gamma) = \emptyset$ if $\sigma \neq \gamma$. Suppose that $(1a)(xb)(y_0c) \in f(\sigma) \cap f(\gamma)$. Then σ has pattern $(1xy_0)(abc)$ and γ has pattern $(1by_0)(axc)$. Therefore the elements in $C_{2,4}$ associated with σ and γ are $(1axb)(y_0c), (1bxc)(y_0a), (1cxa)(y_0b)$ and $(1abx)(y_0c), (1xbc)(y_0a), (1cba)(y_0x)$, respectively. But then we have $(1bxc)(y_0a)$ and $(1abc)(y_0a)$ in the lists, which is a contradiction to a property of \mathcal{Z}_{y_0} . \square

For example, if let $y_0 = 5$, then we have T_4 as follows:

(12)(43)(56)	(13)(46)(52)	(16)(42)(53)	(1 4 5)(2 3 6)	(1 5 4)(2 6 3)
(12)(64)(53)	(13)(62)(54)	(14)(63)(25)	(1 5 6)(2 3 4)	(1 6 5)(2 4 3)
(12)(36)(54)	(14)(32)(65)	(16)(34)(52)	(1 3 5)(2 4 6)	(1 5 3)(2 6 4)
(13)(2 4)(56)	(14)(26)(53)	(16)(23)(54)	(1 2 5)(3 4 6)	(1 5 2)(3 6 4)

Now we define T_3 . For $\mu = (1y_0)(xa')(b'c') \in C_{2,2,2}$, we let

$$f(\mu) = \{\mu, (1b'a'c')(y_0x), (1c'xb')(y_0a'), (1xc'a')(y_0b'), (1a'b'x)(y_0c')\},$$

where $(b'a'c') = (abc)$.

Lemma 2.9. *The set $T_3 = \{f(\mu) : \mu = (1y_0)(**)(**) \in C_{2,2,2}\}$ is a perfect partition of elements in $C_{2,2,2}$ with a 2-cycle $(1y_0)$ and the elements in $C_{2,4}$ in $\mathcal{Z}_{y_0}^\beta$, where $\beta = (1xy_0)(abc)$.*

Proof. We just need to show that $f(\mu) \cap f(\rho) = \emptyset$ if $\mu \neq \rho$. But $(1u'v'w')(y_0x') \in f(\mu) \cap f(\rho)$ only if $\mu = (1y_0)(x'v')(u'w') = \rho$. \square

So with the chosen σ and the pattern, and $y_0 = 5$, we have T_3 as follows:

(1 5)(2 3)(4 6)	(1 2 4 3)(5 6)	(1 3 6 2)(5 4)	(1 4 2 6)(5 3)	(1 6 3 4)(5 2)
(1 5)(2 4)(3 6)	(1 2 6 4)(5 3)	(1 3 4 6)(5 2)	(1 4 3 2)(5 6)	(1 6 2 3)(5 4)
(1 5)(2 6)(3 4)	(1 2 3 6)(5 4)	(1 3 2 4)(5 6)	(1 4 6 3)(5 2)	(1 6 4 2)(5 3)

By the lemmas, we have constructed a perfect partition of $L_{6,1}$, and the conclusion of Theorem 2.1 follows.

3. PERFECT PARTITION $L_{8,2}$

The main theorem of this section is the following.

Theorem 3.1. *The graph $L\{8,2\}$ has a perfect partition.*

We will use the following notation.

- M_n : the set of $n \times n$ real matrices,
- $\{E_{11}, E_{12}, \dots, E_{nn}\}$: standard basis for M_n ,
- $J_n \in M_n$: the matrix with all entries equal to one,
- $O_n \in M_n$: the matrix with all entries equal to zero,
- $C(i, j)$: Swap columns i and j in matrix,
- $R(i, j)$: Swap rows i and j in matrix.

Lemma 3.2. *Suppose a matrix $P \in \mathcal{S}(L_{8,2})$ is written in block form $P = (P_{ij})_{1 \leq i,j \leq 4}$ so that $P_{ij} \in M_2$ for every pair (i, j) . Then either none, one, two, or four of the P_{ij} blocks are invertible, i.e., two of the four entries equal to 1. Thus, $\mathcal{S}(L_{8,2})$ can be partitioned into $\mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_4$, where \mathcal{S}_k consists of matrices $P = (P_{ij})_{1 \leq i,j \leq 4}$ in $\mathcal{S}(L_{8,2})$ such that exactly k of the submatrices P_{ij} are invertible. Moreover, we have:*

$$|\mathcal{S}_0| = 2^8 9, \quad |\mathcal{S}_1| = 2^9 3, \quad |\mathcal{S}_2| = 2^8 3, \quad |\mathcal{S}_4| = 2^4 9.$$

Proof. The set \mathcal{S}_0 contains the matrices for which no blocks are invertible. Then all blocks contain no more than one 1, and each row and column of blocks contain exactly two blocks containing exactly one 1. Denote such a block by E . For the first block column, there are $\binom{3}{2} = 3$ possible choices for which blocks are E . This selection determines that the block not chosen must be O_2 , so the other two non-diagonal entries in that row must be E . The first block row also allows $\binom{3}{2} = 3$ possible choices for which blocks are X , then the other selections of E are determined. Thus, there are $3 \cdot 3 = 9$ combinations of E . Each E may one of $E_{11}, E_{12}, E_{21}, E_{22}$. There are 4 ways to select two E blocks, 2 ways to select the next four E blocks, and 1 way to select the last two. Therefore, $|\mathcal{S}_0| = 2^8 9$.

The set \mathcal{S}_1 contains the matrices for which exactly one block is invertible. In the 2×2 case, this is true only when a block is I_2 or R_2 . Denote such a matrix by X . Then there are 12 non-diagonal positions for which the first X may be placed. All other blocks E and O_2 are determined. The single X may be I_2 or R_2 , so it can be chosen in 2 ways. One E block can be chosen in 4 ways, the next four E blocks can be each chosen in 2 ways, and the last is determined. Thus, $|\mathcal{S}_1| = 2^8 12 = 2^9 3$.

The set \mathcal{S}_2 contains the matrices for which exactly two blocks are invertible. Denote such blocks by X . Then the first X can be placed in one of 12 non-diagonal positions. This placement allows only 2 ways to choose the other X . Since order does not matter, we have $\frac{12 \cdot 2}{2} = 12$ ways to choose the placement of two X blocks. There are two choices for each X

block, 4 choices for the first E block, 2 choices for the next two E blocks, and 1 choice for the last E block. Thus, $|\mathcal{S}_2| = 2^6 12 = 2^8 3$.

The set \mathcal{S}_4 contains the matrices for which exactly four blocks are invertible. Denote those four by X ; then all other blocks must be O_2 . There are 3 ways to place one X in the first block column. Then find the column whose diagonal position is in the same row as the X in the first column. There are 3 ways to place one X in this column. All other X blocks are then determined, so there are $3 \cdot 3 = 9$ ways to place the X blocks. There are 2 ways to choose each X , so $|\mathcal{S}_4| = 2^4 9$. \square

Proof of Theorem 3.1.

We will use the following partitioning scheme for $L_{8,2}$:

Type I. Pick two matrices from \mathcal{S}_0 and four matrices from \mathcal{S}_1 to form subsets.

Type II. Pick four matrices from \mathcal{S}_0 and two matrices from \mathcal{S}_2 to form subsets.

Type III. Pick six matrices from \mathcal{S}_4 to form subsets.

Pick 2 matrices from \mathcal{S}_0 and 4 matrices from \mathcal{S}_1 to form a subset: In the block form, P has 9 perfect matchings which form a perfect partition of 3 subsets of $L_{4,1}$:

$$\begin{aligned} &\{(1,2)(3,4), (1,3,2,4), (1,4,2,3)\}, \\ &\{(1,3)(2,4), (1,2,3,4), (1,4,3,2)\}, \\ &\{(1,4)(2,3), (1,2,4,3), (1,3,4,2)\}. \end{aligned}$$

Let $\mathcal{S}_0^1 \subset \mathcal{S}_0$ be the subset containing all matrices whose non-diagonal zero blocks form a perfect matching of the form $(1,2)(3,4)$, $(1,3)(2,4)$, or $(1,4)(2,3)$.

O	O	E	E
O	O	E	E
E	E	O	O
E	E	O	O

O	E	O	E
E	O	E	O
O	E	O	E
E	O	E	O

O	E	E	O
E	O	O	E
E	O	O	E
O	E	E	O

FIGURE 1. Forms of the elements of \mathcal{S}_0^1 : $(1,2)(3,4)$, $(1,3)(2,4)$, or $(1,4)(2,3)$.

We will use 2 matrices $A_1, A_2 \in \mathcal{S}_0^1$ and 4 matrices $S, T, U, V \in \mathcal{S}_1$ to form the partitions.

Consider a block form $(1, i)(j, k)$ now. Take matrices $P, Q \in \mathcal{S}_0^1$ so that in the sum $A = P + Q$, the blocks $A_{1i}, A_{i1}, A_{jk}, A_{kj}$ are all O_2 , and all other non-diagonal blocks are invertible (here is another way to describe it: choose $E_1 = P_{1j}, E_2 = P_{ik}, E_3 = P_{j1}, E_4 = P_{ki}$ freely, then P and Q are determined to have the desired A).

We shall choose $S, T, U, V \in \mathcal{S}_1$ so that their only invertible blocks are $S_{1i}, T_{i1}, U_{jk}, V_{kj}$ (but we do not know them yet). It follows that the blocks in the same rows and columns as

the invertible ones are all 0s. We also see that $S_{i1}, T_{1i}, U_{kj}, V_{jk}$ must be all 0s, for otherwise, another block must be invertible as well.

Since A_{1j} is invertible and $S_{1j} = V_{1j} = O_2$, $T_{1j} + U_{1j} = J_2 - A_{1j}$; and similarly, $V_{j1} + S_{j1} = J_2 - A_{j1}$. We determine T_{1j} by letting its 1 to be in the same row as P_{1k} and V_{j1} by letting its 1 to be in the same row as P_{ji} . Then, the following blocks are determined, based on the the additional fact that $S + T + U + V = L_{8,2} - A$:

$$(3) \quad T_{1j} \leftrightarrow T_{1k} \leftrightarrow V_{1k} \leftrightarrow V_{ik} \leftrightarrow S_{ik} \leftrightarrow S_{ij} \leftrightarrow U_{ij} \leftrightarrow U_{1j} \leftrightarrow T_{1j}$$

$$(4) \quad V_{j1} \leftrightarrow V_{ji} \leftrightarrow T_{ji} \leftrightarrow T_{ki} \leftrightarrow U_{ki} \leftrightarrow U_{k1} \leftrightarrow S_{k1} \leftrightarrow S_{j1} \leftrightarrow V_{j1}$$

Next, S_{jk} is determined, as its 1 is in different row from S_{j1} (which is determined in (4)) and in different column from S_{ik} (which is determined in (3)). Then T_{jk} and U_{jk} are determined: if $S_{jk} = E_{a,b}$, then $T_{jk} = E_{3-a,3-b}$ and $U_{jk} = J_2 - (S_{jk} + T_{jk})$.

Similarly, we can determine S_{kj}, T_{kj}, V_{kj} ; V_{1i}, U_{1i}, S_{1i} ; and V_{i1}, U_{i1}, T_{i1} . At the end, we will get the matrices S, T, U, V .

Note that we may get S, T, U, V in a similar way by considering T_{jk} first, then S_{jk} and U_{jk} . But it would be the same, since by the chain of determination in (3) and (4), once we know one block in each chain, we know all other blocks, and since we determine T_{jk} or S_{jk} by choosing one block from (3) and (4), there is no way we could get difference results.

From the above process, once we have chosen the pattern for the zero blocks and P_{1j}, P_{j1}, P_{ik} , and P_{ki} , all six matrices are uniquely determined. By symmetry of P and Q , there are $3 \cdot (4 \cdot 4 \cdot 4 \cdot 4 \cdot \frac{1}{2}) = 3 \cdot 2^7$ choices of different pairs $\{P, Q\}$. For each pair $\{P, Q\}$ in \mathcal{S}_0^1 , four different elements of \mathcal{S}_1 are used, so we actually use up all elements in \mathcal{S}_1 .

Pick 4 matrices from $\mathcal{S}_0 - \mathcal{S}_0^1$ and 2 matrices from \mathcal{S}_2 to form a subset: In the block form, P has nine perfect matchings which forms perfect partition of three subsets. In partitions for \mathcal{S}_0^1 and \mathcal{S}_1 , three perfect matchings were used. This partition will use the remaining six. Therefore, no element of \mathcal{S}_0^1 (that was used in the above partition) will be used in this partition.

For a perfect matching $(1, i, j, k)$, (whose inverse is $(1, k, j, i)$), we choose a pair $A_1, A'_1 \in \mathcal{S}_0 - \mathcal{S}_0^1$ so that

- (i) A_1 and A'_1 have the same blocks at $(1, j), (j, 1), (k, i), (i, k)$;
- (ii) the blocks of A_1 at $(1, i), (i, j), (j, k), (k, 1)$ are O_2 ;
- (iii) the blocks of A'_1 at $(1, k), (k, j), (j, i), (i, 1)$ are O_2 .

We get A_2 so that $A_1 + A_2$ only has zero blocks or invertible blocks, and get A_3, A_4 from A_2 by applying operations $\{R(1, 2), R(3, 4), R(5, 6), R(7, 8)\}$ and $\{C(1, 2), C(3, 4), C(5, 6), C(7, 8)\}$, respectively.

Let $B_1, B_2 \in \mathcal{S}_2$ so that $B_1 + B_2 = L_{8,2} - \sum_{i=1}^4 A_i$. Because of the structure of matrices in \mathcal{S}_2 , B_1 and B_2 are determined if the sum is known. Note that S_{jk} is also determined by S_{ij} and S_{k1} . Similarly we get A'_2, A'_3, A'_4 and B'_1, B'_2 from A'_1 .

Now we show that every matrix in $\mathcal{S}_2 \cup (\mathcal{S}_0 - \mathcal{S}_0^1)$ appears exactly once in the above construction. Note that for each choices of blocks at positions $(1, j), (j, 1), (k, i), (i, k)$, we get two different partitions of $L_{8,2}$, with eight matrices in \mathcal{S}_0 and four in \mathcal{S}_2 . We can partition the matrices in $\mathcal{S}_0 - \mathcal{S}_0^1$ into sets of eight matrices, and each set uses four matrices in \mathcal{S}_2 . So in total $\frac{1}{2} \cdot |\mathcal{S}_0 - \mathcal{S}_0^1| = 3 \cdot 2^8$ matrices in \mathcal{S}_2 are used, that is, we use up all matrices in \mathcal{S}_2 .

The perfect partitions using only matrices in \mathcal{S}_4 : In the block form, P has 9 perfect matchings which form a perfect partition with three subsets. Each block J_2 in the perfect matchings can be decomposed into two invertible submatrices I_2 and R_2 , so we will have six matrices from \mathcal{S}_4 summing to $L_{8,2}$.

For each of the subsets, we can apply one of the 7 operations $\{R(1,2)\}, \{R(3,4)\}, \{R(5,6)\}, \{R(7,8)\}, \{R(1,2), R(3,4)\}, \{R(1,2), R(5,6)\}, \{R(1,2), R(7,8)\}$ to get a different partition. In such a way, we use up all $18 \cdot 8 = 9 \cdot 2^4$ matrices in \mathcal{S}_4 to form perfect partitions. \square

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